1 $y = 2x^3 + 5x^2 - 7x + 10$

(a) Find $\frac{dy}{dx}$ **(2)**

(b) Find the gradient of the curve when $x = 2$ **(1)**

a/ $\frac{dy}{dx} = 6x^2 + 10x - 7$

b/ when $x = 2$ $\frac{dy}{dx} = 6(2)^2 + 10(2) - 7$

$= 37$

---

2 $y = 3x + \frac{1}{x}$

(a) Find $\frac{dy}{dx}$ **(2)**

(b) Find the x coordinates of points where the gradient is zero. **(2)**

a/ $y = 3x + x^{-1}$

$\frac{dy}{dx} = 3 - x^{-2}$

b/ $3 - x^{-2} = 0$

$3 = x^{-2}$

$3 = \frac{1}{x^2}$

$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}}$

---

3 $f(x) = 3x^{\frac{3}{2}} + \frac{3}{x^2} - 6x$

Find $f'(x)$

$f(x) = 3x^{\frac{3}{2}} + 3x^{-2} - 6x$

$f'(x) = \frac{9}{2}x^{\frac{1}{2}} - 6x^{-3} - 6$

4 $y = 4\sqrt{x} + \dfrac{1}{2x} + 10$

(a) Find $\dfrac{dy}{dx}$ **(3)**

(b) Find $\dfrac{d^2y}{dx^2}$ **(2)**

$$y = 4x^{\frac{1}{2}} + \frac{1}{2}x^{-1} + 10$$

a/ $$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - \frac{1}{2}x^{-2}$$

b/ $$\frac{d^2y}{dx^2} = -x^{-\frac{3}{2}} + x^{-3}$$

---

5 $y = \dfrac{2x^2 - 5x + 3}{x}$

(a) Find $\dfrac{dy}{dx}$ **(3)**

(b) Find the gradient when $x = 3$ **(1)**

$$y = 2x - 5 + 3x^{-1}$$

a/ $$\frac{dy}{dx} = 2 - 3x^{-2}$$

b/ when $x = 3$ $\quad \dfrac{dy}{dx} = 2 - 3(3)^{-2}$

$$= \frac{5}{3}$$

---

6 $y = x^3 - 4x^2 - 3x + 9$

(a) Find $\dfrac{dy}{dx}$ **(2)**

(b) Find the range values of x for which y is increasing **(3)**

a/ $$\frac{dy}{dx} = 3x^2 - 8x - 3$$

b/ $$3x^2 - 8x - 3 > 0$$

$$\underline{\underline{x < -\frac{1}{3}}} \quad \text{or} \quad \underline{\underline{x > 3}}$$

7 A curve has the equation $y = 2x^3 + 9x^2 - 24x + 13$
Find the coordinates of the curve's local maximum.

$$\frac{dy}{dx} = 6x^2 + 18x - 24$$

turning point where $\frac{dy}{dx} = 0$

$$6x^2 + 18x - 24 = 0$$
$$x^2 + 3x - 4 = 0$$
$$(x+4)(x-1) = 0$$
$$x = -4 \qquad x = 1$$

$$\frac{d^2y}{dx^2} = 12x + 18$$

when $x = -4$ $\quad \frac{d^2y}{dx^2} = -30$ (maximum)

$x = 1$ $\quad \frac{d^2y}{dx^2} = 30$ (minimum)

$$y = 2(-4)^3 + 9(-4)^2 - 24(-4) + 13$$
$$= 125$$

$$\underline{\underline{(-4, 125)}}$$

---

8 $$y = 4x^2 + \frac{16}{x} + 1 \qquad x > 0$$

(a) Find $\frac{dy}{dx}$ **(3)**

(b) Find the exact range of values of x for which the curve is increasing. **(2)**

a/ $$y = 4x^2 + 16x^{-1} + 1$$

$$\frac{dy}{dx} = 8x - 16x^{-2}$$

b/ $$8x - 16x^{-2} > 0$$
$$8x - \frac{16}{x^2} > 0$$
$$8x^3 - 16 > 0$$
$$x^3 > 2$$
$$x > \sqrt[3]{2}$$

9 A curve has the equation $y = 2x^3 - 12x^2 + 18x + 5$

(a) The curve has a local minimum at P, find the coordinates of P. **(4)**

(b) Justify that P is a minimum point. **(2)**

a/ $\frac{dy}{dx} = 6x^2 - 24x + 18$

Minimum where $\frac{dy}{dx} = 0$

$6x^2 - 24x + 18 = 0$

$x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$x = 3 \qquad x = 1$

$\frac{d^2y}{dx^2} = 12x - 24$

when $x = 3$ $\quad \frac{d^2y}{dx^2} = 12$ $\quad$ positive $\therefore$ minimum.

$y = 2(3)^3 - 12(3)^2 + 18(3) + 5$

$= 5$

$\underline{\underline{(3, 5)}}$

b/ $\frac{d^2y}{dx^2} = 12x - 24$

when $x = 3$ $\quad \frac{d^2y}{dx^2} = 12$ $\quad$ positive $\therefore$ minimum.

10 A curve has the equation $y = 3x^2 - 5x + 7$

Find the equation of the tangent to the curve at the point $P(2, 9)$.

Write your answer in the form $y = mx + c$, where m and c are integers to be found.

$$\frac{dy}{dx} = 6x - 5$$

when $x = 2$ $\quad \frac{dy}{dx} = 6(2) - 5$

$= 7 \qquad m = 7 \qquad \underset{x\ \ y}{(2, 9)}$

$$y - 9 = 7(x - 2)$$
$$y - 9 = 7x - 14$$
$$y = 7x - 5$$

---

11 A curve has the equation $y = g(x)$

Given that
- $g(x)$ is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x
- the curve with equation $y = g(x)$ passes through the origin
- the curve with equation $y = g(x)$ has a stationary point at $(2, -10)$

(a) Find $g(x)$ **(7)**

(b) prove that the stationary point at (2, -10) is a minimum. **(2)**

$c = 0$

$$g(x) = ax^3 + bx^2 + ax + c$$
$$g(x) = ax^3 + bx^2 + ax$$
$$g'(x) = 3ax^2 + 2bx + a$$

stationary point when $x = 2$ $\quad 3a(2)^2 + 2b(2) + a = 0$

$12a + 4b + a = 0$

$13a + 4b = 0$ (2)

$-10 = a(2)^3 + b(2)^2 + a(2)$

$-10 = 8a + 4b + 2a$

$10a + 4b = -10$ (1)

$a = \frac{10}{3} \qquad b = \frac{-65}{6}$

$$g(x) = \frac{10}{3}x^3 - \frac{65}{6}x^2 + \frac{10}{3}$$

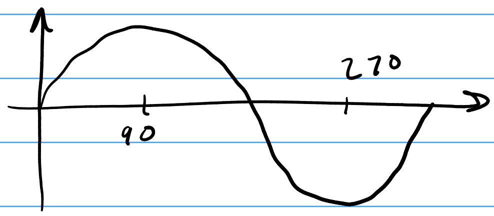
b/ $g''(x) = 6ax + 2b$

$= 20x - \frac{65}{3}$

$g''(2) = 20(2) - \frac{65}{3} = \frac{55}{3}$

positive $\therefore$ minimum

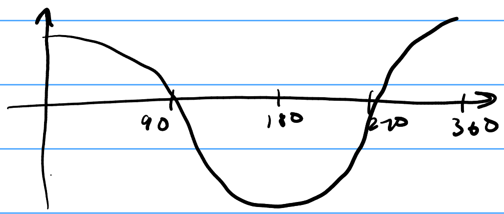
12 State the interval for which $y = \sin x$ is a decreasing function for $0° \leq x \leq 360°$



decreasing when the gradient is negative

$90 < x < 270$

13 State the interval for which $y = \cos x$ is an increasing function for $0° \leq x \leq 360°$



$180 < x < 360$

14 $y = 2x^3 + 5x^2 - 7x + 10$

Find the equation of the tangent at the point where $x = 1$
Give your answer in the form $y = mx + c$

$$\frac{dy}{dx} = 6x^2 + 10x - 7$$

when $x = 1$ $\quad \frac{dy}{dx} = 6(1)^2 + 10(1) - 7$

$$= 9 \qquad m = 9$$

$$y = 2(1)^3 + 5(1)^2 - 7(1) + 10$$

$$= 10 \qquad (1, 10)$$

$$y - 10 = 9(x - 1)$$

$$= 9x - 9$$

$$y = 9x + 1$$

15 $f(x) = 2x^3 + x^2 - 18x + 2$

The points A and B lie on the curve $y = f(x)$. The gradient at both A and B is 2.
Find the coordinates of A and B.

$f'(x) = 6x^2 + 2x - 18$

$2 = 6x^2 + 2x - 18$

$0 = 6x^2 + 2x - 20$

$0 = 3x^2 + x - 10$

$0 = (3x - 5)(x + 2)$

$x = \frac{5}{3}$ $\quad$ $x = -2$

$y = 2\left(\frac{5}{3}\right)^3 + \left(\frac{5}{3}\right)^2 - 18\left(\frac{5}{3}\right) + 2$ $\quad$ $\left(\frac{5}{3}, \frac{-431}{27}\right)$

$y = 2(-2)^3 + (-2)^2 - 18(-2) + 2$ $\quad$ $(-2, 26)$

---

16 $y = \dfrac{(4x-1)(x+2)}{2x}$

Find the equation of the normal at the point when $x = -2$
Give your answer in the form $ax + by + c = 0$ where a, b and c are integers.

$y = \dfrac{4x^2 + 8x - x - 2}{2x}$

$= \dfrac{4x^2 + 7x - 2}{2x}$

$= 2x + \frac{7}{2} - x^{-1}$

$\frac{dy}{dx} = 2 + x^{-2}$

when $x = -2$

$y = \dfrac{(4(-2) - 1)(-2+2)}{2(-2)}$

$= 0$

$(-2, 0)$
$x_1 \quad y_1$

when $x = -2$ $\quad$ $\frac{dy}{dx} = 2 + (-2)^{-2}$

$= \frac{9}{4}$

perp $m = -\frac{4}{9}$

$y - 0 = -\frac{4}{9}(x + 2)$

$9y = -4x - 8$

$4x + 9y + 8 = 0$

17 A simple model for the cost of a car journey £C when a car is driven at a steady speed of v mph is

$$C = \frac{4500}{v} + v + 10$$

(a) Use this model to find the value of v which minimises the cost of the journey. **(5)**

(b) Use $\frac{d^2C}{dv^2}$ to verify that C is a minimum for this value of v **(2)**

(c) Calculate the minimum cost of the journey **(2)**

a/ $C = 4500v^{-1} + v + 10$

$\frac{dC}{dv} = -4500v^{-2} + 1$

$-4500v^{-2} + 1 = 0$

$1 = \frac{4500}{v^2}$

$v^2 = 4500$

$v = 67.1$ mph

b/ $\frac{d^2C}{dv^2} = 9000v^{-3}$

when $v = 67.1$ $\frac{d^2C}{dv^2} = 0.0298$ +ve $\therefore$ Minimum

c/ $C = \frac{4500}{67.1} + 67.1 + 10$

$= £144$ (3sf)

18 A cylinder has a radius r and a height h.
The surface area of the cylinder is 500cm^2

(a) Show that the volume (V cm^3) of the cylinder is given by $V = 250r - \pi r^3$ **(4)**

Given that r varies

(b) Calculate the maximum value of V, to the nearest cm^3 **(6)**

(c) Justify that the value of V you found is a maximum. **(2)**

a/ volume $= \pi r^2 h$

$s.a = 2\pi r^2 + 2\pi rh$

$500 = 2\pi r^2 + 2\pi rh$

$250 = \pi r^2 + \pi rh$

$250 - \pi r^2 = \pi rh$

$h = \dfrac{250 - \pi r^2}{\pi r}$

volume $= \pi r^2 \left(\dfrac{250 - \pi r^2}{\pi r} \right)$

$= \dfrac{250\pi r^2 - \pi^2 r^4}{\pi r}$

$= \underline{\underline{250r - \pi r^3}}$

b/ $\dfrac{dV}{dr} = 250 - 3\pi r^2$

max when $250 - 3\pi r^2 = 0$

$250 = 3\pi r^2$

$\dfrac{250}{3\pi} = r^2$

$r = \sqrt{\dfrac{250}{3\pi}}$

$= \underline{\underline{5.15}}$

$v = 250(5.15) - \pi(5.15)^3$

$= \underline{\underline{858 \text{ cm}^3}}$

c/ $\dfrac{d^2v}{dr^2} = -6\pi r$ when $r = 5.15$

$\dfrac{d^2v}{dr^2} = -97.1$ <u>negative $\therefore$ maximum</u>

19 A curve has the equation $y = 4x^3 + 15x^2 - 18x + 5$

Find the coordinates of the stationary points and determine the nature of each stationary point.

$$\frac{dy}{dx} = 12x^2 + 30x - 18$$

stationary points where $\frac{dy}{dx} = 0$

$$12x^2 + 30x - 18 = 0$$

$$x = \frac{1}{2} \qquad x = -3$$

$$y = \frac{1}{4} \qquad y = 86$$

$$\left(\frac{1}{2}, \frac{1}{4}\right) \qquad (-3, 86)$$

$$\frac{d^2y}{dx^2} = 24x + 30$$

when $x = \frac{1}{2}$ $\frac{d^2y}{dx^2} = 42$ positive $\therefore$ minimum at $\left(\frac{1}{2}, \frac{1}{4}\right)$

when $x = -3$ $\frac{d^2y}{dx^2} = -42$ negative $\therefore$ maximum at $(-3, 86)$

20 A curve has equation $y = 3x^2 - 16x\sqrt{x} + 18x - 2$ for $x \geq 0$

(a) Prove that the curve has a maximum point at (1, 3) **(9)**
Fully justify your answer.

(b) Find the coordinates of the other stationary point of the curve and state its nature. **(2)**

$$y = 3x^2 - 16x^{\frac{3}{2}} + 18x - 2$$

$$\frac{dy}{dx} = 6x - 24x^{\frac{1}{2}} + 18$$

Maximum point when $\frac{dy}{dx} = 0$

$$6x - 24x^{\frac{1}{2}} + 18 = 0$$
$$x - 4x^{\frac{1}{2}} + 3 = 0$$
$$(x^{\frac{1}{2}} - 3)(x^{\frac{1}{2}} - 1) = 0$$

$$x^{\frac{1}{2}} = 3 \qquad x^{\frac{1}{2}} = 1$$
$$\underline{x = 9} \qquad \underline{x = 1}$$

$$y = 3(1)^2 - 16(1)^{3/2} + 18(1) - 2$$
$$\underline{= 3}$$

$\underline{(1, 3)}$

$$\frac{d^2y}{dx^2} = 6 - 12x^{-\frac{1}{2}}$$

when $x = 1$ $\quad \frac{d^2y}{dx^2} = 6 - 12(1)^{-\frac{1}{2}}$

$$= -6$$

<u>negative $\therefore$ minimum</u>

b/ $y = 3(9)^2 - 16(9)^{3/2} + 18(9) - 2$
$= -29$

$\underline{(9, -29)}$ $\quad \frac{d^2y}{dx^2} = 6 - 12(9)^{-\frac{1}{2}} = 2$ $\quad$ positive $\therefore$ <u>minimum</u>

21 A company is designing a cup. The cup will be in the shape of a cylinder with radius x and height h.

The cup does not have a lid and must hold 450 ml of liquid.

(a) Show that the surface area of the cup is given by $\pi x^2 + \dfrac{900}{x}$ **(4)**

(b) Find, to 2 decimal places, the value of x that makes the surface area a minimum. **(4)**

(c) Justify that the value of x you found is a minimum. **(2)**

(d) Give a reason why the company may not choose to make a cup with a radius this size. **(1)**

volume $= \pi r^2 h$

$450 = \pi r^2 h$

$\dfrac{450}{\pi r^2} = h$

$s.a = \pi r^2 + 2\pi r h$

$= \pi r^2 + 2\pi r \cdot \dfrac{450}{\pi r^2}$

$= \pi r^2 + \dfrac{900}{r}$

$= \pi x^2 + \dfrac{900}{x}$ $(r = x)$

b/ $\dfrac{dS}{dx} = 2\pi x - 900x^{-2}$

min where $\dfrac{dS}{dx} = 0$ $2\pi x - \dfrac{900}{x^2} = 0$

$2\pi x = \dfrac{900}{x^2}$

$2\pi x^3 = 900$

$x^3 = \dfrac{900}{2\pi}$

$x = 5.23$ (cm)

c/ $\dfrac{d^2S}{dx^2} = 2\pi + 1800x^{-3}$

when $x = 5.23$ $\dfrac{d^2S}{dx^2} = 6\pi$ positive $\therefore$ minimum

d/ It would be too wide. It may spill easily or be difficult to hold.

22 Prove that the curve with equation

$$y = 4x^5 + 15x^4 + 20x^3 + 7$$

only has one stationary point, stating its coordinates.

$$\frac{dy}{dx} = 20x^4 + 60x^3 + 60x^2$$

stationary point where $\frac{dy}{dx} = 0$

$$20x^4 + 60x^3 + 60x^2 = 0$$

$$x^4 + 3x^3 + 3x^2 = 0$$

$$x^2(x^2 + 3x + 3) = 0$$

$x = 0$ or $x^2 + 3x + 3 = 0$

$b^2 - 4ac < 0$

$(3)^2 - 4(1)(3) = -3$

$\therefore$ no solutions

when $x = 0$ $y = 4(0)^5 + 15(0)^4 + 20(0)^3 + 7$

$= 7$

$(0, 7)$

23 A curve has equation

$$y = 2x^3 - 3x^2 + 4x - 5$$

(a) Find $\frac{dy}{dx}$ **(4)**

(b) Show that the perpendicular bisector of the line joining A(6, 2) and B(4, -6) is a normal to the curve at (1, -1). **(6)**

a/ $\frac{dy}{dx} = 6x^2 - 6x + 4$

b/ when $x = 1$ $\quad \frac{dy}{dx} = 4 \quad (1, -1)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_1 \quad y_1$

$\therefore m = -\frac{1}{4}$

$$y + 1 = -\frac{1}{4}(x - 1)$$

$$4y + 4 = -(x - 1)$$

$$4y + 4 = -x + 1$$

$$\underline{\underline{x + 4y + 3 = 0}}$$

Midpoint of AB $= \left(\frac{6+4}{2}, \frac{2-6}{2}\right)$

$= (5, -2)$

$\quad x_1 \quad y_1$

$$m_{AB} = \frac{-6-2}{4-6} = \frac{-8}{-2} = 4$$

perp. $m = -\frac{1}{4}$

$$y + 2 = -\frac{1}{4}(x - 5)$$

$$4y + 8 = -(x - 5)$$

$$4y + 8 = -x + 5$$

$$\underline{\underline{x + 4y + 3 = 0}}$$

24 A curve has equation

$$y = x^3 + px^2 + qx - 5$$

The curve passes through the point A (2, 1)

$x \quad y$

The gradient of the curve at A is 5.

Find the value of p and the value of q.

$1 = (2)^3 + p(2)^2 + q(2) - 5$

$1 = 8 + 4p + 2q - 5$

$-2 = 4p + 2q$

$-1 = 2p + q \quad (1)$

$\frac{dy}{dx} = 3x^2 + 2px + q \qquad$ when $x = 2 \quad \frac{dy}{dx} = 5$

$5 = 3(2)^2 + 2p(2) + q$

$5 = 12 + 4p + q$

$-7 = 4p + q \quad (2)$

$p = -3 \qquad q = 5$

25 A curve has equation $f(x) = (x + 3)(x - 2)^2$

(a) Find the coordinates of the turning points of the curve.
Determine the nature of each turning point. **(8)**

(b) State the coordinates of the turning points of the curve $y = 2f(x - 1)$ **(2)**

a/ $f(x) = (x+3)(x^2 - 4x + 4)$

$= x^3 + 3x^2 - 4x^2 - 12x + 4x + 12$

$= x^3 - x^2 - 8x + 12$

$f'(x) = 3x^2 - 2x - 8$

turning points where $f'(x) = 0$

$3x^2 - 2x - 8 = 0$

$(3x+4)(x-2) = 0$

$x = -\frac{4}{3}$ $\quad x = 2$

$y = \frac{500}{27}$ $\quad y = 0$

$\left(-\frac{4}{3}, \frac{500}{27}\right)$ $\quad (2, 0)$

$f''(x) = 6x - 2$

when $x = -\frac{4}{3}$ $\quad f''(x) = -10$ $\quad f''(x) < 0$

$\therefore$ Local Maximum

when $x = 2$ $\quad f''(x) = 10$ $\quad f''(x) > 0$

$\therefore$ Local Minimum.

26 A curve has equation $y = 3x^4 - 2\sqrt{x} + \frac{x}{2} - 2$

Find an expression for $\frac{d^2y}{dx^2}$

$$y = 3x^4 - 2x^{\frac{1}{2}} + \frac{1}{2}x - 2$$

$$\frac{dy}{dx} = 12x^3 - x^{-\frac{1}{2}} + \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = 36x^2 + \frac{1}{2}x^{-3/2}$$

---

27 A curve has equation $y = ax^2 - \frac{b}{\sqrt{x}} + \frac{c}{x}$

(a) In terms of a, b and c, find an expression for $\frac{dy}{dx}$ **(4)**

(b) In terms of a, b and c, find an expression for $\frac{d^2y}{dx^2}$ **(3)**

$$y = ax^2 - bx^{-\frac{1}{2}} + cx^{-1}$$

a/ $$\frac{dy}{dx} = 2ax + \frac{1}{2}x^{-\frac{3}{2}} - cx^{-2}$$

b/ $$\frac{d^2y}{dx^2} = 2a - \frac{3}{4}x^{-\frac{5}{2}} + 2cx^{-3}$$

28 A curve has equation $y = x^2 - 4x$

(a) Find $\frac{dy}{dx}$ **(2)**

(b) Find the values of x for which y is increasing. **(2)**

a/ $\frac{dy}{dx} = 2x - 4$

b/ $2x - 4 > 0$

$2x > 4$

$x > 2$

---

29 The line $y = 3x + k$ is a tangent to the curve $x^2 - y = 3$. Find the value of the constant k.

one solution to sim. eq.

$x^2 - (3x + k) = 3$

$x^2 - 3x - k = 3$

$x^2 - 3x - k - 3 = 0$

$b^2 - 4ac = 0$

$(-3)^2 - 4(1)(-k-3) = 0$

$9 - 4(-k - 3) = 0$

$9 + 4k + 12 = 0$

$4k + 21 = 0$

$k = \frac{-21}{4}$

30 Find the equation of the normal to the curve $y = 2\sqrt{x} + 3x + 1$ at the point where $x = 4$.

Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$$y = 2x^{\frac{1}{2}} + 3x + 1$$

$$\frac{dy}{dx} = x^{-\frac{1}{2}} + 3$$

when $x = 4$ $\quad \frac{dy}{dx} = (4)^{-\frac{1}{2}} + 3$

$$= \frac{7}{2}$$

gradient of normal $= -\frac{2}{7}$ m

when $x = 4$ $\quad y = 2\sqrt{4} + 3(4) + 1$

x $\quad = 17$

y

$$y - 17 = -\frac{2}{7}(x - 4)$$

$$7(y - 17) = -2(x - 4)$$

$$7y - 119 = -2x + 8$$

$$\underline{\underline{2x + 7y - 127 = 0}}$$

31 Find the equation of the normal to the curve $y = (2x - 1)^2$ at the point where $x = 2$.

Give your answer in the form $y = mx + c$

$$y = 4x^2 - 4x + 1$$

$$\frac{dy}{dx} = 8x - 4$$

when $x = 2$ $\quad \frac{dy}{dx} = 12$

$\therefore \quad m = -\frac{1}{12}$

when $x = 2$ $\quad y = (2(2) - 1)^2$

$\quad = 9$

$$y - 9 = -\frac{1}{12}(x - 2)$$

$$y - 9 = -\frac{1}{12}x + \frac{1}{6}$$

$$\underline{\underline{y = -\frac{1}{12}x + \frac{55}{6}}}$$

32 (a) Sketch the gradient function of the curve $y = x^3 - 3x^2 - 45x$ **(5)**

(b) Determine the set of values for which $x^3 - 3x^2 - 45x$ is decreasing **(2)**

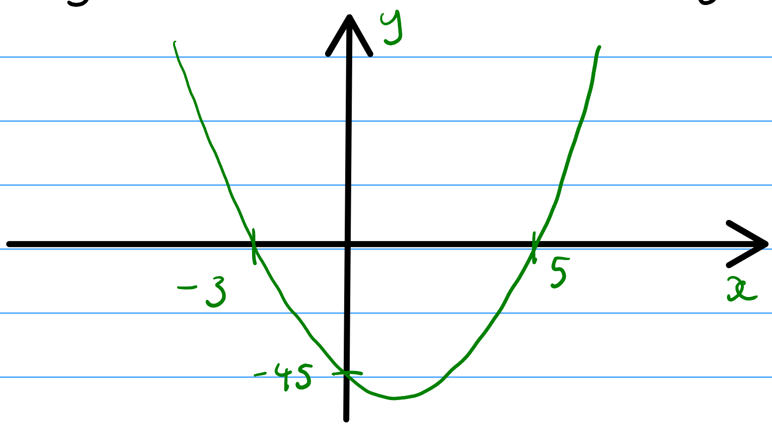
a) $\frac{dy}{dx} = 3x^2 - 6x - 45$

sketch $y = 3x^2 - 6x - 45$

crosses x when $y = 0$

$$0 = 3x^2 - 6x - 45$$
$$0 = x^2 - 2x - 15$$
$$0 = (x-5)(x+3)$$
$$x = 5 \quad x = -3$$

crosses y when $x = 0$ $\quad y = -45$



b) $x^3 - 3x^2 - 45x$ is decreasing where the gradient function is below the x-axis.

$$\underline{-3 < x < 5}$$

33 The equation of a curve is $y = 2x^2 + \frac{1}{x}$

A tangent and a normal to the curve are drawn at the point where $x = 1$.

Calculate the area bounded by the tangent, the normal and the x-axis.

when $x = 1$ $\quad y = 2(1)^2 + \frac{1}{1}$

$= 3$

$$y = 2x^2 + x^{-1}$$

$$\frac{dy}{dx} = 4x - x^{-2}$$

when $x = 1$ $\quad \frac{dy}{dx} = 3$

gradient of tangent $= 3$

gradient of normal $= -\frac{1}{3}$

Tangent

$y - 3 = 3(x - 1)$

$y - 3 = 3x - 3$

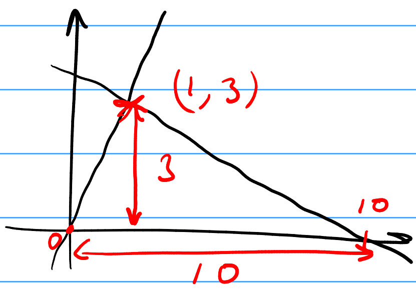
$y = 3x$

Normal

$y - 3 = -\frac{1}{3}(x - 1)$

$y - 3 = -\frac{1}{3}x + \frac{1}{3}$

$y = -\frac{1}{3}x + \frac{10}{3}$



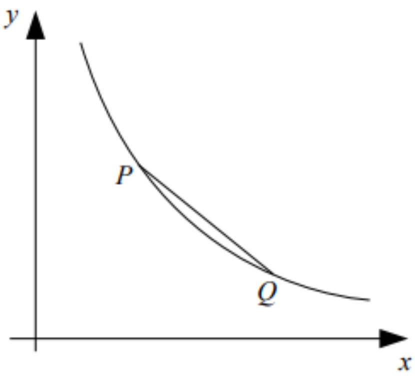
lines cross x when $y = 0$

$x = 0$ and $x = 10$

Area $= \frac{1}{2} \times 10 \times 3$

$= 15$ units^2

34 The graph shows part of the curve with equation $y = \frac{4}{x}$



P is the point with coordinates (1, 4) and Q is the point with x coordinate $(1+ h)$

The table shows for different values of h, the coordinates of P, the coordinates of Q and the gradient of the chord PQ

x for P	y for P	h	x for Q	y for Q	Gradient
1	4	1	2	2	-2
1	4	0.1	1.1	3.636364	-3.636364
1	4	0.01	1.01	3.960396	-3.960396
1	4	0.001	1.001	3.996004	-3.996004

(a) Complete the table. **(3)**

(b) Explain how the sequence of values in the last column relates to the gradient of the curve at the point P. **(1)**

(c) Use calculus to find the gradient of the curve at the point P. **(2)**

b/ The gradient is getting closer and closer to the gradient at P.

c/ $y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2}$

when $x = 1$ $\quad \frac{dy}{dx} = -4(1)^{-2}$

$= -4$

35 Danny is investigating the gradient of chords of the curve with equation $f(x) = 2 - x^2$

Each chord joins the point (3, -7) to the point $(3 + h, f(3 + h))$

The table shows Danny's results.

x	$f(x)$	h	$x + h$	$f(x + h)$	Gradient
3	-7	1	4	-14	-7
3	-7	0.1	3.1	-7.61	-6.1
3	-7	0.01	3.01	-7.0601	-6.01
3	-7	0.001	3.001	-7.006001	-6.001

(a) Complete the table. **(5)**

(b) Suggest the limit for the gradient of these chords as h tends to 0. **(1)**

b/ -6

36 A cuboid *ABCDEF* has width $2x$, height x and depth y.

The volume of the cuboid is 600 cm³. The surface area of the cuboid is S cm².

(a) Show that $S = 4x^2 + \dfrac{1800}{x}$ **(5)**

(b) Determine the value of x for which the surface area of the cuboid is a minimum. **(4)**

(c) Find, to the nearest integer, the minimum value of S. **(1)**

a/

$$\text{volume} = 2x \cdot x \cdot y$$

$$600 = 2x^2 y \longrightarrow y = \frac{600}{2x^2} = \frac{300}{x^2}$$

$$S = 2(2x^2) + 2(2xy) + 2(xy)$$

$$= 4x^2 + 4xy + 2xy$$

$$= 4x^2 + 6xy$$

$$S = 4x^2 + 6x\left(\frac{300}{x^2}\right)$$

$$= 4x^2 + \frac{1800}{x}$$

b/

$$\frac{dS}{dx} = 8x - 1800x^{-2}$$

$$8x - \frac{1800}{x^2} = 0$$

$$8x^3 - 1800 = 0$$

$$x^3 = 225$$

$$x = 6.08 \text{ cm}$$

c/

$$S = 4(6.08)^2 + \frac{1800}{6.08}$$

$$= 444 \text{ cm}^2$$

37 (i) A curve has equation $y = 8x + \frac{1}{2x^2}$

(a) Find an expression for $\frac{dy}{dx}$ **(2)**

(b) Find an expression for $\frac{d^2y}{dx^2}$ **(2)**

(ii) Hence find the coordinates of the stationary point and determine its nature. **(5)**

$$y = 8x + \frac{1}{2}x^{-2}$$

a/ $$\frac{dy}{dx} = 8 - x^{-3}$$

b/ $$\frac{d^2y}{dx^2} = 3x^{-4}$$

ii/ $$8 - x^{-3} = 0$$

$$8 - \frac{1}{x^3} = 0$$

$$8 = \frac{1}{x^3}$$

$$x^3 = \frac{1}{8}$$

$$x = \frac{1}{2}$$

$$y = 8\left(\frac{1}{2}\right) + \frac{1}{2\left(\frac{1}{2}\right)^2}$$

$$= 6$$

Stationary point at $\left(\frac{1}{2}, 6\right)$

$$\frac{d^2y}{dx^2} = 3\left(\frac{1}{2}\right)^{-4} = 48$$

+ve $\therefore$ local minimum

38 Show that the only stationary point on the graph of $y = 2x^2 - 8\sqrt{x}$ is a minimum point at (1, -6)

$$y = 2x^2 - 8x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4x - 4x^{-\frac{1}{2}}$$

stationary point where $\frac{dy}{dx} = 0$

$$4x - 4x^{-\frac{1}{2}} = 0$$

$$x - x^{-\frac{1}{2}} = 0$$

$$x^{\frac{3}{2}} - 1 = 0$$

$$x^{\frac{3}{2}} = 1$$

$$\underline{\underline{x = 1}}$$

when $x = 1$ $y = 2(1)^2 - 8(\sqrt{1})$

$= -6$

$$\underline{\underline{(1, -6)}}$$

$$\frac{d^2y}{dx^2} = 4 + 2x^{-\frac{3}{2}}$$

when $x = 1$ $\frac{d^2y}{dx^2} = 6$

positive $\therefore$ <u>Minimum</u>

39 Prove, from first principles, that the derivative of $4x$ is 4.

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = 4x$

$f(x+h) = 4(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{4(x+h) - 4x}{h}$$

$$= \lim_{h \to 0} \frac{4x + 4h - 4x}{h}$$

$$= \lim_{h \to 0} \frac{4h}{h}$$

$$= \lim_{h \to 0} 4$$

$$= \underline{\underline{4}}$$

---

40 Prove, from first principles, that the derivative of x^3 is $3x^2$.

$f(x) = x^3$ $\quad$ $f(x+h) = (x+h)^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x^2+2xh+h^2) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + hx^2 + 2x^2h + 2xh^2 + xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= \underline{\underline{3x^2}}$$

41 Prove, from first principles, that the derivative of $2x^3$ is $6x^2$.

$f(x) = 2x^3 \qquad f(x+h) = 2(x+h)^3$

$= 2(x+h)(x^2+2xh+h^2)$

$= 2(x^3+x^2h+2x^2h+2xh^2+xh^2+h^3)$

$= 2(x^3+3x^2h+3xh^2+h^3)$

$= 2x^3+6x^2h+6xh^2+2h^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2x^3+6x^2h+6xh^2+2h^3-2x^3}{h}$$

$$= \lim_{h\to 0} \frac{6x^2h+6xh^2+2h^3}{h}$$

$$= \lim_{h\to 0} 6x^2+6xh+2h^2$$

$$= \underline{\underline{6x^2}}$$

---

42 Prove, from first principles, that the derivative of $5x^2$ is $10x$.

$f(x) = 5x^2 \qquad f(x+h) = 5(x+h)^2$

$= 5(x^2+2xh+h^2)$

$= 5x^2+10xh+5h^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5x^2+10xh+5h^2-5x^2}{h}$$

$$= \lim_{h\to 0} \frac{10xh+5h^2}{h}$$

$$= \lim_{h\to 0} 10x+5h$$

$$= \underline{\underline{10x}}$$

43 Prove, from first principles, that the derivative of kx^3 is $3kx^2$.
Where k is a constant.

$f(x) = kx^3 \qquad f(x+h) = k(x+h)^3$

$= k(x+h)(x^2+2xh+h^2)$

$= k(x^3 + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3)$

$= k(x^3 + 3x^2h + 3xh^2 + h^3)$

$= kx^3 + 3kx^2h + 3kxh^2 + kh^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{kx^3 + 3kx^2h + 3kxh^2 + kh^3 - kx^3}{h}$$

$$= \lim_{h \to 0} \frac{3kx^2h + 3kxh^2 + kh^3}{h}$$

$$= \lim_{h \to 0} 3kx^2 + 3kxh + kh^2$$

$$= \underline{\underline{3kx^2}}$$

44 A curve C has equation $y = 3x^2 + 1$

The point $P(3, 28)$ lies on the curve.

(a) Find the gradient of the tangent at P. **(2)**

The point Q with x-coordinate $(3 + h)$ also lies on C.

(b) Find the gradient of the line PQ, giving your answer in terms of h in its simplest form. **(3)**

(c) Explain briefly the relationship between part (b) and the answer to part (a). **(1)**

a/ $\dfrac{dy}{dx} = 6x$

when $x = 3$ $\quad \dfrac{dy}{dx} = 6(3) = \underline{\underline{18}}$

b/ $(3, 28)$ $\quad$ $x_1 \quad y_1$

when $x = 3 + h$ $\quad x_2$

$y = 3(3+h)^2 + 1$

$= 3(9 + 6h + h^2) + 1$

$= 27 + 18h + 3h^2 + 1$

$= 3h^2 + 18h + 28$ $\quad y_2$

$$m = \frac{3h^2 + 18h + 28 - 28}{3 + h - 3}$$

$$= \frac{3h^2 + 18h}{h}$$

$$= 3h + 18$$

c/ As h gets closer to zero the gradient gets closer to 18 (the answer to (a)).

45 Differentiate $3x^2 + x$ from first principles.

$f(x) = 3x^2 + x$ $\qquad f(x+h) = 3(x+h)^2 + (x+h)$

$$= 3(x^2 + 2xh + h^2) + x + h$$

$$= 3x^2 + 6xh + 3h^2 + x + h$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + x + h - (3x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + x + h - 3x^2 - x}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 + h}{h}$$

$$= \lim_{h \to 0} 6x + 3h + 1$$

$$= \underline{\underline{6x + 1}}$$

46 Differentiate $4x - 3x^2$ from first principles.

$$f(x) = 4x - 3x^2 \qquad f(x+h) = 4(x+h) - 3(x+h)^2$$

$$= 4x + 4h - 3(x^2 + 2xh + h^2)$$

$$= 4x + 4h - 3x^2 - 6xh - 3h^2$$

$$f'(x) = \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6xh - 3h^2 - (4x - 3x^2)}{h}$$

$$= \lim_{h \to 0} \frac{4x + 4h - 3x^2 - 6xh - 3h^2 - 4x + 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{4h - 6xh - 3h^2}{h}$$

$$= \lim_{h \to 0} 4 - 6x - 3h$$

$$= 4 - 6x$$

47 (a) Sketch the gradient function of the curve $y = x^3 + 3x^2 - 24x$ **(5)**

(b) Determine the set of values for which $x^3 + 3x^2 - 24x$ is increasing **(2)**

a/ $\frac{dy}{dx} = 3x^2 + 6x - 24$

sketch $y = 3x^2 + 6x - 24$

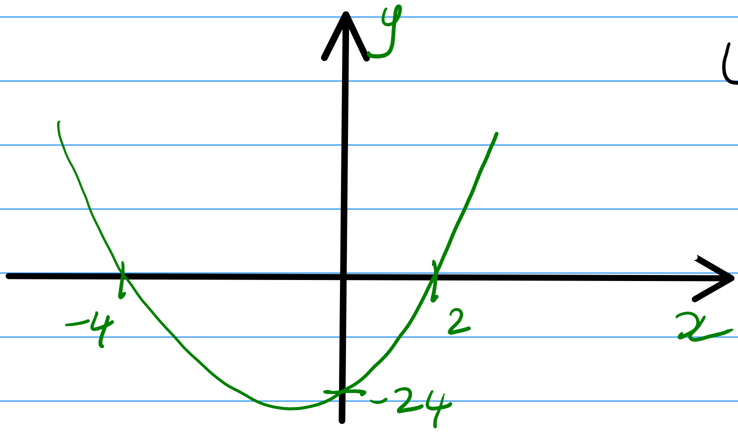
crosses y when $x = 0$ $\quad y = -24$

crosses x when $y = 0$ $\quad 3x^2 + 6x - 24 = 0$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4 \quad x = 2$



b/ Increasing when $\underline{\underline{x < -4}}$ or $\underline{\underline{x > 2}}$

48 A curve has equation $y = x^3 + x^2$
A normal to the curve is drawn at the point where $x = 1$ and meets the x-axis at A and the y-axis at B.

(a) Find the area of OAB. **(6)**

(b) Use calculus to prove the curve has one local maximum and one local minimum point. **(6)**

a/ when $x = 1$ (x_1) $\quad y = (1)^3 + (1)^2 = 2$ (y_1)

$\frac{dy}{dx} = 3x^2 + 2x$

when $x = 1$ $\quad \frac{dy}{dx} = 5$ $\quad$ (normal) $\therefore m = -\frac{1}{5}$

$y - 2 = -\frac{1}{5}(x - 1)$

$5(y-2) = -(x-1)$

$5y - 10 = -x + 1$

crosses y when $x = 0$

$5y - 10 = 1$

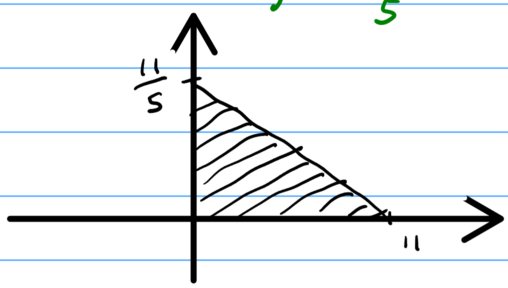
$5y = 11$

$y = \frac{11}{5}$

crosses x when $y = 0$

$-10 = -x + 1$

$x = 11$



Area $= \frac{1}{2} \times 11 \times \frac{11}{5}$

$= \frac{121}{10}$ units^2

b/ $\frac{dy}{dx} = 3x^2 + 2x$

turning points where $\frac{dy}{dx} = 0$

$3x^2 + 2x = 0$

$x(3x + 2) = 0$

$x = 0 \quad x = -\frac{2}{3}$

$\frac{d^2y}{dx^2} = 6x + 2$

when $x = 0$ $\quad \frac{d^2y}{dx^2} = 2$

positive $\therefore$ local minimum

when $x = -\frac{2}{3}$ $\quad \frac{d^2y}{dx^2} = -2$

negative $\therefore$ local maximum

49 The equation of a curve is $y = 4\sqrt{x} - 8x^2$

(a) Find $\frac{dy}{dx}$ **(3)**

(b) Find the coordinates of the turning point. **(3)**

(c) Determine the nature of the turning point. **(2)**

a/ $y = 4x^{\frac{1}{2}} - 8x^2$

$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - 16x$

b/ $2x^{-\frac{1}{2}} - 16x = 0$

$2x^{-\frac{1}{2}} = 16x$

$x^{-\frac{1}{2}} = 8x$

$1 = 8x^{3/2}$

$\frac{1}{8} = x^{\frac{3}{2}}$

$\frac{1}{64} = x^3$

$\underline{\underline{x = \frac{1}{4}}}$

$y = 4\sqrt{\frac{1}{4}} - 8\left(\frac{1}{4}\right)^2$

$= \frac{3}{2}$

$\underline{\underline{\left(\frac{1}{4}, \frac{3}{2}\right)}}$

$\frac{d^2y}{dx^2} = -x^{-\frac{3}{2}} - 16$

when $x = \frac{1}{4}$ $\frac{d^2y}{dx^2} = -24$

negative $\therefore$ $\underline{\underline{\text{maximum}}}$